GAPS OF A CLASS OF PSEUDO SYMMETRIC NUMERICAL SEMIGROUPS

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ABSTRACT. In this study, we give some results about the gaps, fundamental and special gaps of a pseudo symmetric numerical semigroup in the form of $S = <3, 3 + s, 3 + 2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$.

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and nonnegative integers, respectively. A numerical semigroup is a subset $S$ of $\mathbb{N}$ that is closed under addition where $0 \in S$ and $\mathbb{N}\setminus S$ is finite. It is well known that every numerical semigroup is finitely generated [1], that is to say, there exist $s_1, s_2, ..., s_p \in \mathbb{N}$ such that $s_1 < s_2 < ... < s_p$ and

$$S = < s_1, s_2, ..., s_p > = \{ s_1k_1 + s_2k_2 + ... + s_pk_p : k_i \in \mathbb{N}, 1 \leq i \leq p \}.$$ Moreover, every numerical semigroup has a unique minimal system of generators.

Following the notation used in [2,3], if $S$ is a numerical semigroup then the greatest integer in $\mathbb{Z}\setminus S$ is the Frobenius number of $S$, denoted by $g(S)$. The elements of $\mathbb{N}\setminus S$, denoted by $H(S)$ are called gaps of $S$. If $x \in H(S)$ and $\{2x, 3x\} \subset S$ then $x$ is called the fundamental gap. We denote by $FH(S)$ the set of fundamental gaps of $S$. $S$ is symmetric if for every $x \in \mathbb{Z}\setminus S$, the integer $g(S) - x \in S$. Similarly, $S$ is pseudo symmetric if $g(S)$ is even and there exists an integer $x \in \mathbb{Z}\setminus S$ such that $x = \frac{g(S)}{2}$ and $g(S) - x \notin S$. For more background on symmetric and pseudo symmetric numerical semigroups, the reader is encouraged to see [2,3,4,7,9].

Let $S$ be a numerical semigroup and $m \in S\setminus\{0\}$. The Apéry set of $S$ with respect to $m$ is defined by $Ap(S, m) = \{ s \in S : s - m \notin S \}$. Hence, $Ap(S, m) = \{ w(0) = 0, w(1), w(2), ..., w(m - 1) \}$ and $g(S) = \max(Ap(S, m)) - m$, where $w(i)$ is the least element in $S$ that is congruent with $i$ modulo $m$. For instance see [6] and [10].
The following can be found in [7]: Let $S$ be a numerical semigroup. We say that $x \in \mathbb{Z}\setminus S$ is a pseudo Frobenius number of $S$ if $x + s \in S$ for all $s \in S\setminus \{0\}$. We denote by $Pg(S)$ the set of pseudo Frobenius numbers of $S$. The cardinal of $Pg(S)$ is called the type of $S$ and denoted by $\text{type}(S)$. Notice that $g(S)$ is always an element of $Pg(S)$. In [11], it is proved that a numerical semigroup is symmetric if and only if $Pg(S) = \{g(S)\}$ i.e. $\text{type}(S) = 1$. Furthermore, we define in $S$ the following partial order:

$$a \leq_S b \text{ if } b - a \in S.$$

For $m \in S\setminus \{0\}$, it is proved that $Pg(S) = \{w(i) - m : w(i)\text{maximal } \leq_S Ap(S,m)\}$ in [7].

An element $x \in Pg(S)$ is a special gap of $S$ if $2x \in S$. We denote by $SH(S)$ the set of special gaps of $S$. That is, $SH(S) = \{x \in Pg(S) : 2x \in S\}$. The following proposition is proved in [8]:

$$x \in Pg(S) \text{ if and only if } S \cup \{x\} \text{ is a numerical semigroup.}$$

The main goal of this paper is to prove Theorem 2 and Theorem 3 which gives the sets $H(S)$ and $FH(S)$ with respect to $s$. We also find the cardinality $\sharp(FH(S))$ and give the relations between $\sharp(H(S))$ and $\sharp(FH(S))$ in Corollary 4 and Corollary 5.

In this paper, $S$ is defined as $S = < 3, 3+s, 3+2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$.

2. Results

In this section, we will give some results related to the gaps, fundamental and special gaps of a pseudo symmetric numerical semigroup in the form $S = < 3, 3+s, 3+2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$.

Firstly we give following theorem:

**Theorem 1.** $S = < 3, 3+s, 3+2s >$ is a pseudo symmetric numerical semigroup, for $s \in \mathbb{Z}^+$ and $3 \nmid s$. [see 5,9].

**Notation:** We can write the following cases for $S$:

(i) If $s = 6k + 1$ or $s = 6k + 4$ then

$$S = < 3, 3+s, 3+2s > = \{0, 3, ..., s-1, s+2, s+3, s+5, ..., 2s+1, \to \ldots\}$$

(ii) If $s = 6k + 2$ or $s = 6k + 5$ then

$$S = < 3, 3+s, 3+2s > = \{0, 3, ..., s-2, s+1, s+3, s+4, ..., 2s+1, \to \ldots\}$$
where \( k \in \mathbb{N} \).

**Theorem 2.** The set of gaps of \( S \) is as follows:

(i) if \( s = 6k + 1 \) or \( s = 6k + 4 \), then

\[
H(S) = \{1, 2, 4, 5, \ldots, s, s + 1, s + 4, \ldots, 2s\}.
\]

(ii) if \( s = 6k + 2 \) or \( s = 6k + 5 \), then

\[
H(S) = \{1, 2, 4, 5, \ldots, s, s + 2, s + 5, s + 8, \ldots, 2s\}
\]

where \( k \in \mathbb{N} \).

**Proof.** By definition, every non-positive integer \( k \) with \( k \leq s \), \( 3 \nmid k \) is in \( H(S) \). That is, \( \{1, 2, 4, 5, \ldots, s\} \subseteq H(S) \). In addition, for the different states of \( s \):

(i) If \( s = 6k + 1 \) (\( k \in \mathbb{N} \)) then \( 3 \nmid (s + 1) \), so \( s + 1 \in H(S) \). However, \( s + 2, s + 3 \in S \). In this case, \( s + 1 + 3t \leq 2s \) \((t \in \mathbb{N})\). Otherwise, let \( s + 1 + 3t \notin H(S) \) for \( s + 1 + 3t \leq 2s \), then \( s + 1 + 3t \in S \). Thus, \( 3 \mid (s + 1) \) since \( 3 \mid (s + 1 + 3t) \) that is \( 3 \mid (6k + 2) \). This is a contradiction. Therefore, \( H(S) = \{1, 2, 4, 5, \ldots, s, s + 1, s + 4, \ldots, 2s\} \).

If \( s = 6k + 4 \), then \( 3 \nmid s \), but \( s + 2, s + 3 \in S \). That is \( s + 1 \in H(S) \). On the contrary, let \( s + 1 \notin H(S) \). Then \( 3 \mid s + 1 \) and \( 3 \mid 6k + 5 \) which is a contradiction. Thus, \( s + 1 + 3t \in H(S) \) is obtained for \( s + 1 + 3t \leq 2s \). Consequently, \( H(S) = \{1, 2, 4, 5, \ldots, s, s + 1, s + 4, \ldots, 2s\} \).

(ii) If \( s = 6k + 2 \), then \( 3 \mid s + 1 \) and \( s + 1, s + 3, s + 4 \in S \); but \( s + 2 \notin S \). In other words, \( s + 2 \in H(S) \). We assume that \( s + 2 \notin H(S) \). Then, \( 3 \mid s + 2 \), that is \( 3 \mid 6k + 4 \). Hence, \( 3 \mid 4 \) which gives a contradiction. Thus, we have that \( H(S) = \{1, 2, 4, 5, \ldots, s, s + 2, s + 5, s + 8, \ldots, 2s\} \).

If \( s = 6k + 5 \) then \( s + 1, s + 3 \in S \). But \( s + 2 \notin S \), i.e. \( s + 2 \in H(S) \). Conversely, \( s + 2 \notin H(S) \). Then \( 3 \mid s + 2 \) and \( 3 \mid 6k + 7 \) which is a contradiction. Hence, \( s + 2 + 3t \in H(S) \) for \( s + 1 + 3t \leq 2s \). Thus, \( H(S) = \{1, 2, 4, 5, \ldots, s, s + 2, s + 5, s + 8, \ldots, 2s\} \) is obtained.

**Theorem 3.** The set of fundamental gaps of \( S \) is given as follows:

(a) if \( s = 6k + 1 \) or \( s = 6k + 5 \), then \( FH(S) = \left\{ \frac{3 + s}{2}, \frac{3 + s}{2} + 3, \ldots, 2s \right\} \)

(b) if \( s = 6k + 2 \) or \( s = 6k + 4 \), then \( FH(S) = \left\{ \frac{6 + s}{2}, \frac{6 + s}{2} + 3, \ldots, 2s \right\} \)

where \( k \in \mathbb{N} \).
Proof. (a) We must firstly show that \( T = \left\{ \frac{3+s}{2}, \frac{3+s}{2} + 3, \ldots, 2s \right\} \neq \emptyset \) and \( T \subseteq H(S) \). Thus it suffices to prove \( \frac{3+s}{2} \notin S \) (since \( n = \frac{3+s}{2} \in H(S) \) for \( \frac{3+s}{2} \notin S \) and \( n + 3t \leq 2s \) \( (t \in \mathbb{N}) \), \( n, 3t \in H(S) \)). Conversely, assume that \( \frac{3+s}{2} \in S \). In this case, \( \frac{3+s}{2} = 3n + (3+s)n_2 + (3+2s)n_3 \) \((n_1, n_2, n_3 \in \mathbb{N})\). Thus, we write \( s = 3(2n_1 - 1) + (3+s)2n_2 + (3+2s)2n_3 \in S \). But this yields \( s \in S \) which contradicts with the definition of \( S \). Now let us show that \( T = FH(S) \):

\[
x \in T \implies x = \frac{3+s}{2} + 3t, \ (t \in \mathbb{N})
\]

\[
\implies 2x = 2\left(\frac{3+s}{2} + 3t\right) \text{ and } 3x = 3\left(\frac{3+s}{2} + 3t\right)
\]

\[
\implies 2x = 3 + s + 6t \text{ and } [3x = 3\left(\frac{3+6k+1}{2} + 3t\right) \text{ or } 3x = 3\left(\frac{3+6k+5}{2} + 3t\right)]
\]

\[
\implies 2x \in S \text{ and } [3x = 6 + 9k + 9t \text{ or } 3x = 12 + 9k + 9t]
\]

\[
\implies 2x \in S \text{ and } 3x \in S
\]

\[
\implies x \in FH(S).
\]

For the other implication, let us show that \( FH(S) \subseteq T \). Conversely, assume that \( FH(S) \notin T \). Then, \( \exists y \in FH(S) \implies y \notin T \), i.e., \( y \notin H(S) \), which gives \( y \in S \). This is a contradiction. As a result \( FH(S) = T \).

(b) \( A = \left\{ \frac{6+s}{2}, \frac{6+s}{2} + 3, \ldots, 2s \right\} \) is a subset of \( H(S) \): For this, it suffices to prove \( \frac{6+s}{2} \notin S \) (since \( v = \frac{6+s}{2} \in H(S) \) for \( \frac{6+s}{2} \notin S \), and \( v + 3t \leq 2s \) \( (t \in \mathbb{N}) \), \( v, 3t \in H(S) \)). Conversely, assume that \( \frac{6+s}{2} \in S \). In this case, \( \frac{6+s}{2} = 3u_1 + (3+s)u_2 + (3+2s)u_3 \) \((u_1, u_2, u_3 \in \mathbb{N})\). Thus, we write \( s = 3(2u_1 - 2) + (3+s)2u_2 + (3+2s)2u_3 \in S \). This contradicts with the definition of \( S \). Furthermore, \( T = FH(S) \):

\[
x \in T \implies x = \frac{6+s}{2} + 3t, \ (t \in \mathbb{N})
\]

\[
\implies 2x = 2\left(\frac{6+s}{2} + 3t\right) \text{ and } 3x = 3\left(\frac{6+s}{2} + 3t\right)
\]

\[
\implies 2x = 6 + s + 6t \text{ and } [3x = 3\left(\frac{6+6k+2}{2} + 3t\right) \text{ or } 3x = 3\left(\frac{6+6k+4}{2} + 3t\right)]
\]

\[
\implies 2x \in S \text{ and } [3x = 12 + 9k + 9t \text{ or } 3x = 15 + 9k + 9t]
\]

\[
\implies 2x \in S \text{ and } 3x \in S
\]

\[
\implies x \in FH(S).
\]

On the other hand, \( FH(S) \subseteq T \) can be shown as in (a).

**Corollary 4.**

(i) If \( s \) is odd, then \( \sharp(FH(S)) = \frac{s+1}{2} \).

(ii) If \( s \) is even, then \( \sharp(FH(S)) = \frac{s}{2} \).
Proof. By Theorem 3, we have that $FH(S) = \{ \frac{3+s}{2}, \frac{3+s}{2} + 3, \ldots, 2s \}$ and $FH(S) = \left\{ \frac{6+s}{2}, \frac{6+s}{2} + 3, \ldots, 2s \right\}$ are obtained where $s$ is odd and even, respectively. Thus, if $s$ is odd, then $\sharp(FH(S)) = \frac{2s-\frac{3+s}{3}}{6} + 1 = \frac{3s-3}{6} + 1 = \frac{s+1}{2}$. If $s$ is even, then $\sharp(FH(S)) = \frac{2s-\frac{6+s}{2}}{6} + 1 = \frac{3s-6}{6} + 1 = \frac{s}{2}$.

**Corollary 5.** The following corollary a result of Corollary 4
(i) If $s$ is odd, then $\sharp(H(S)) = 2\sharp(FH(S))$.
(ii) If $s$ is even, then $\sharp(H(S)) = 2\sharp(FH(S)) + 1$.

**Proposition 6.** The set of special gaps of $S$ is $\{2s\}$, that is, $SH(S) = \{2s\}$.

Proof. We can write that $Ap(S, 3) = \{0, 3 + s, 2s + 3\}$ and

$$Maximals \leq_S (Ap(S, 3)) = \left\{ \frac{2s}{2} + 3, 2s + 3 \right\}$$

from [5] and [9], respectively. Thus, we write that

$$SH(S) = \{x \in Pg(S) : 2x \in S\}$$

since $Pg(S) = \{s, 2s\}$.

**Corollary 7.** $SH(S) \subset FH(S) \subset H(S)$.

**Example 8.** Let $S = \langle 3, 7, 11 \rangle = \{0, 3, 6, 7, 9, 10, 11, \ldots\}$ be a pseudo symmetric numerical semigroup for $s = 4$. Since $s = 4 = 6 + 4; g(S) = 8, Ap(S, 3) = \{0, 3 + 4, 2.4 + 3\} = \{0, 7, 11\}$, and $H(S) = \{1, 2, 4, 5, 8\}, FH(S) = \left\{ \frac{6+4}{2}, 6+4 + 3 \right\} = \{5, 8\}, SH(S) = \{8\}$.

Thus, $\sharp(H(S)) = 4 + 1 = 5 = 2\sharp(FH(S)) + 1$ and $\{8\} \subset \{5, 8\} \subset \{1, 2, 4, 5, 8\}$.

**References**


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