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ON A CLASS OF PSEUDO-SYMMETRIC NUMERICAL SEMIGROUPS

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Abstract
In this paper, we give some results on a class of pseudo-symmetric
numerical semigroups such that $S = (3, 3 + s, 3 + 2s)$, where $s \in \mathbb{Z}^+$
and 3 is not the multiplicity of $s$.

1. Introduction
A numerical semigroup $S$ is a subset of $\mathbb{N}$ (the set of nonnegative integers)
containing 0 and closed under addition such that $\mathbb{N} \setminus S$ is a finite set.

For a numerical semigroup $S, A = \{s_1, s_2, \ldots, s_p\} \subset S$ is a generating set of $S$.

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provided that $S = \langle s_1, s_2, ..., s_p \rangle = \{k_1 s_1 + k_2 s_2 + \cdots + k_p s_p : k_i \in \mathbb{N}, 1 \leq i \leq p \}$. The set $A = \{s_1, s_2, ..., s_p\}$ is called minimal generating set of $S$ if no proper subset $A'$ generating set of $S$. It was observed in [1] that the set $\mathbb{N} \setminus S$ is finite if and only if $\gcd\{s_1, s_2, ..., s_p\} = 1$.

Another important invariant of $S$ is the largest integer not belonging to $S$ known as the Frobenius number of $S$ denoted by $g(S)$, that is, we have $g(S) = \max\{x \in \mathbb{Z} : x \in S\}$ [1]. We define $n(S) = \#((0, 1, 2, ..., g(S)) \cap S)$, where $\#(A)$ denotes the cardinality of a set $A$. It is also well-known that $S = \{0, s_1, s_2, ..., s_{n-1}\}$, $s_n = g(S) + 1$, $\rightarrow$ means that every integer greater than $g(S) + 1$ belongs to $S$, $n = n(S)$ and $s_i < s_{i+1}$, for $i = 1, 2, ..., n$.

The elements of $\mathbb{N} \setminus S$ denoted by $H(S)$ also, are called the gaps of $S$. A gap $x$ of a numerical semigroup $S$ is said to be fundamental if $\{2x, 3x\} \subseteq S$. We denote the set of all fundamental gaps of $S$ by $FH(S)$ [5]. $S$ is a symmetric if for each $x \in \mathbb{Z} \setminus S$, the integer $g(S) - x$ belongs to $S$. Moreover, a numerical semigroup $S$ is pseudo-symmetric if $g(S)$ is even and $x = \frac{g(S)}{2}$ is the only integer such that $x \in \mathbb{Z} \setminus S$ and $g(S) \neq x \in S$ [3].

For $m \in S \setminus \{0\}$, the Apéry set of $m$ in $S$ is $A_p(S, m) = \{s \in S : s - m \notin S\}$. It can easily be proved that $A_p(S, m)$ is formed by the smallest elements of $S$ belonging to the different congruence classes mod $m$. According to this, we can write $\#(A_p(S, m)) = m$ and $g(S) = \max(A_p(S, m)) - m$. Various aspects and properties of Apéry sets are detailed in [6]. Let $S$ be a numerical semigroup and $n$ be a positive integer. Then $\mathbb{S}_n = \{x \in \mathbb{N} : nx \in S\}$ is also a numerical semigroup containing $S$ and it is called the quotient of $S$ by $n$ [8].

For a numerical semigroup $S = \langle s_1, s_2, ..., s_p \rangle$, the Hilbert series of $S$ is given by

$$H(S : x) = \frac{1}{1 - x^{s_1}} \sum_{a \in A_p(S, s_1)} x^a. \quad [4]$$

In this paper, we give some results about a class of pseudo-symmetric numerical semigroups generated by the set $\langle 3, 3 + s, 3 + 2s \rangle$, where $s \in \mathbb{Z}^+$ and 3 is not the multiplicity of $s$.

2. Results

In this section, we give some results about a class of pseudo-symmetric numerical semigroups generated by three elements $3, 3 + s, 3 + 2s$ such that $s \in \mathbb{Z}^+$ and 3 is not the multiplicity of $s$.

**Theorem 2.1.** $S = \langle 3, 3 + s, 3 + 2s \rangle$ is a pseudo-symmetric numerical semigroup, where $s \in \mathbb{Z}^+$ and 3 is not the multiplicity of $s$.

**Proof.** $S = \langle 3, 3 + s, 3 + 2s \rangle$ is a numerical semigroup since 3 is not the multiplicity of $s$ and $\gcd(3, 3 + s) = 1$.

We put $n_1 = 3, n_2 = s + 3$ and $n_3 = 3 + 2s$. Let us define the numbers

$$c_i = \min\{x \in \mathbb{N}^+ : x n_i \in \langle n_1, n_2, n_3 \rangle \setminus \{n_1\}\}$$

for $i = 1, 2, 3$. If we show following equalities, then proof is completed from [2]:

$$c_1 n_1 = (c_2 - 1) n_2 + n_3,$$

$$c_2 n_2 = (c_3 - 1) n_3 + n_1,$$

$$c_3 n_3 = (c_1 - 1) n_1 + n_2.$$

That $c_1 = \min\{x \in \mathbb{N}^+ : 3x \in \langle 3 + s, 2s + 3 \rangle\} = 2$ because if $3x \in \langle 3 + s, 3 + 2s \rangle$, then there exist $k, r \in \mathbb{N}^+$ such that $3x = (3 + s)k + (3 + 2s)r$. If we put $k = r = 1$ to find the minimum $x$, then $3x = (3 + s) + (3 + 2s) = 3s + 6 = 3(s + 2)$. Thus

$$x = s + 2$$

and $c_1 = s + 1$,

$$c_2 = \min\{x \in \mathbb{N}^+ : (3 + s)x \in \langle 3, 2s + 3 \rangle\} = 2.$$

If $(3 + s)x \in \langle 3, 3 + 2s \rangle$, then there exist $m, n \in \mathbb{N}^+$ such that $(3 + s)x = 3m + (3 + 2s)n$. Similarly, we put $m = n = 1$ for the minimum $x$ and we have

$$(3 + s)x = 3 + (3 + 2s) = 2s + 6 = 2(s + 3).$$

Thus, it follows that $x = 2$ and $c_2 = 2$. 

At last, $c_3 = 2$. Because, $(3 + 2s)x \in (3, 3 + s)$, and hence there exist $k_1, k_2 \in \mathbb{N}^+$ such that $(3 + 2s)x = 3k_1 + (3 + s)k_2$. In this case, $k_2 > 1$ is not true. If $k_2 > 1$, then we put $k_2 = 2$ and we have $(3 + 2s)x = 3k_1 + (3 + s)2 = 3k_1 + (6 + 2s) = 3k_1 + 3(3 + 2s)$. Thus, we cannot find any $x \in \mathbb{N}^+$. Therefore, $k_2 = 1$. Hence, the least positive integer such that $(3 + 2s)x = 3k_1 + (3 + s)$ is obtained to be 2. Otherwise, if $x = 1$, then $s = 3k_1$ from $(3 + 2s) = 3k_1 + (3 + s)$. This is a contradiction by hypothesis. Finally, $c_3 = 2$.

For $c_1 = s + 2$, $c_2 = c_3 = 2$, we get the following equalities:

\[
c_1n_1 = (s + 2)3 = (2 - 1)(s + 3) + (3 + 2s) = (c_2 - 1)n_2 + n_3,
\]

\[
c_2n_2 = 2(s + 3) = (2 - 1)(3 + 2s) + (3) = (c_3 - 1)n_3 + n_1,
\]

\[
c_3n_3 = 2(3 + 2s) = (s + 1)3 + (s + 3) = (s + 2 - 1)3 + (s + 3) = (c_1 - 1)n_1 + n_2.
\]

**Corollary 2.2.** Let $S = (3, 3 + s, 3 + 2s)$ be a pseudo-symmetric numerical semigroup, where $s \in \mathbb{Z}^+$ and 3 be not the multiplicity of s. Then $g(S) = 2s$ and $Ap(S, 3) = \{0, s + 3, 3 + 2s\}$.

**Proof.** If we put $a = s + 2$, $b = c = 2$ in [2, Theorem 14], then we obtain that $g(S) = 2((s + 2) - 1)(2 - 1) - 2 = 2s$. On the other hand, $\#(H(S)) = g(S) + 1 = s + 1$, since $S$ is pseudo-symmetric [3]. Thus

$Ap(S, 3) = \{0 < 3(s + 1) - 2s < 2s + 3\} = \{0, s + 3, 2s + 3\}$, from [7] and $g(S) = \max(Ap(S, 3)) - 3$.

**Corollary 2.3.** Let $S$ be a pseudo-symmetric numerical semigroup as in Theorem 2.1, and $s$ be an even positive integer. Then $\frac{S}{2} = \left(3, 3 + \frac{s}{2}, s + 3\right)$ is also a pseudo-symmetric numerical semigroup and $g\left(\frac{S}{2}\right) = s$.

**Proof.** Let $T = \left(3, 3 + \frac{s}{2}, s + 3\right)$. Then $T$ is numerical semigroup since $\gcd\left(3, 3 + \frac{s}{2}\right) = 1$. If $x \in T$, then there exist $t_1, t_2, t_3 \in \mathbb{N}$ such that $x = 3t_1 + \left(\frac{6 + s}{2}\right)t_2 + (3 + s)t_3$. Thus

\[
2x = 2(3t_1 + (6 + s)t_2 + (6 + 2s)t_3) = 3(2t_1 + (3 + s)t_2 + 2t_3 + (3 + 2s)t_3) + 3t_3 = 3(2t_1 + t_2 + t_3) + (3 + s)t_2 + (3 + 2s)t_3.
\]

Hence, we obtain $x \in \frac{S}{2}$.

Now, let $a = \frac{S}{2}$. Then $2a \in S = (3, 3 + s, 2s + 3)$. Hence, there exist $m_1, m_2, m_3 \in \mathbb{N}$ such that $2a = 3m_1 + (3 + s)m_2 + (3 + 2s)m_3$. In this case, we have

\[
a = \frac{3m_1}{2} + \left(\frac{3 + s}{2}\right)m_2 + \left(\frac{3 + 2s}{2}\right)m_3
\]

\[
= \frac{3m_1}{2} + \left(\frac{3m_2}{2}\right) + \frac{sm_2}{2} + \frac{sm_3}{2} + \frac{3m_3}{2}
\]

\[
= \left(\frac{m_1 - m_2 - m_3}{2}\right) + \left(\frac{3 + s}{2}\right)m_2 + (3 + s)m_3,
\]

where the most two's of $m_1, m_2$ and $m_3$ must be odd since $s$ is even. Therefore, we obtain that $a \in T$. On the other hand, from Corollary 2.2 and [3], we write

$g(S) = 2s$ and $g\left(\frac{S}{2}\right) = g\left(\frac{S}{2}\right) = s$, respectively. Thus, $g\left(\frac{S}{2}\right) = s$.

**Corollary 2.4.** Let $S = (3, 3 + s, 3 + 2s)$ be a pseudo-symmetric numerical semigroup, where $s \in \mathbb{Z}^+$ and 3 be not the multiplicity of $s$. Then the Hilbert series of $S$ is given by

\[
H(S : x) = \frac{x^{2s+3} + x^{s+3} + 1}{1 - x^3}
\]

and the degree of $H(S : x)$ is $2s$.
Proof. It is clear that from [4] and Corollary 2.2.

Example 2.5. For $s = 8$, $S = \{3, 11, 19\} = \{0, 3, 6, 9, 11, 12, 14, 15, 17, 18, 19, \ldots\}$ is a pseudo-symmetric numerical semigroup. Then, Frobenius number of $S$ is $\alpha(S) = 16$. The set of gaps and the set of fundamental gaps of $S$, $H(S) = \{1, 2, 4, 5, 7, 8, 10, 13, 16\}$ and $F(s) = \{7, 10, 13, 16\}$, respectively. Thus, we find that

$$F(S, 3) = \{0, s + 3, 2s + 3\} = \{0, 11, 19\}, \quad \frac{S}{2} = \{x \in \mathbb{N} : 2x \in S\} = \{3, 3 + \frac{S}{2}, s + 3\} = \{3, 7, 11\} = \{0, 3, 6, 7, 9, 10, 11, \ldots\}$$

is pseudo-symmetric numerical semigroup and $\alpha(S) = 8$. Finally, the Hilbert series of $S$ is

$$H(S : x) = \frac{x^{19} + x^{11} + 1}{1 - x^3}$$

and its degree is 16.

References


